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Convergence Acceleration of Iterative Modal Reduction Methods

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An accelerated method is presented for the iterative condensation of eigenproblems. The present study was motivated by the improved reduction system and the succession-level approximate reduction (SAR). The reduction procedures are supplemented with the second-order approximation in the series expansion of the system transformation. The reduced equation of an equivalent system and the transformation matrix are updated in an iterative manner. In addition, systematic derivation and comparison of the equations involved in various condensations have been sought. The matrix update incorporates not only inverse iteration but also subspace transformation implicitly. The series expansion can be considered as repeated updates of the transformation matrix through inverse iteration. The solution accuracy is sensitive to the selection of the degrees of freedom, for which sequential elimination or energy method may be preferable. When a poor selection causes a sudden failure of the update method, the hybrid dynamic condensation can be used. The method of SAR is closely related to the hybrid dynamic condensation.

Nomenclature

$[A_s]$	=	$[k_{ss}]^{-1}[m_{ss}]$ in Eq. (10)
$[B_{pp}]$, $[B_{sp}]$	=	submatrices for inertia force in Eq. (68)
$[D_G]$	=	$[M_G]^{-1}[K_G]$ dynamic matrix in Eq. (13)
$[K_e]$, $[M_e]$	=	equivalent structural matrices in Eq. (26)
$[K_G]$, $[M_G]$	=	reduced matrices in Guyan's condensation
$[K_r]$, $[M_r]$	=	structural matrices reduced through $[T_r]$
$[K_X]$, $[M_X]$	=	structural matrices reduced through $[T_X]$
$[k]$, $[k_{\alpha\beta}]$	=	stiffness matrix and submatrices
$[m]$, $[m_{\alpha\beta}]$	=	mass matrix and submatrices
p	=	number of primary degrees of freedom
$[Q_{pp}]$	=	modal matrix for generalized coordinates
$\{R\}$	=	residual error in eigenproblem
s	=	number of secondary degrees of freedom
$[T]$	=	exact transformation matrix in Eq. (22)
$[T_i]$	=	matrices in series of Eqs. (15) and (18)
$[T_p]$	=	system transformation matrix
$[T_r]$	=	approximate transformation matrix
$[T_X]$	=	improved transformation matrix in Eq. (40)
$[T_0]$	=	transformation in Guyan's condensation
$[T(\lambda)]$	=	frequency-dependent transformation in Eq. (3)
$[X_p]$	=	improved modal matrix
$[X_{pp}]$, $[X_{sp}]$	=	submatrices of $[X_p]$
ε	=	convergence tolerance
$[\Lambda_p]$	=	diagonal matrix for p eigenvalues
λ, λ_{app}	=	eigenvalues, exact and approximate
λ_s	=	lowest eigenvalue of secondary subspace
$[\Phi_p]$, $[\Phi_p]_{\text{app}}$	=	p mode shapes, exact and approximate
$[\Phi_{pp}]$, $[\Phi_{sp}]$	=	submatrices of $[\Phi_p]$
$\{\phi\}$, $\{\phi\}_{\text{app}}$	=	mode shapes, exact and approximate
$\{\phi_p\}$, $\{\phi_p\}_{\text{app}}$	=	primary sets, exact and approximate
$\{\phi_s\}$, $\{\phi_s\}_{\text{app}}$	=	secondary sets, exact and approximate
$[\Psi_p]$	=	orthogonalized form of $[X_p]$
$[\Psi_{pp}]$, $[\Psi_{sp}]$	=	submatrices of $[\Psi_p]$

Superscripts

(i)	=	iteration number
T	=	transpose of matrix
-1	=	inverse of matrix

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Introduction

SIGNIFICANT computational resources and time are required for the dynamic analysis of large structural systems. Ever-increasing computer capabilities and efficient numerical algorithms have made it feasible to obtain an almost exact solution. For general users who have limited computing facilities, however, various reduction methods are an alternative to get the lowest eigenmodes with accuracy acceptable in an engineering sense.

In system condensation, primary (master) degrees of freedom are retained in the analysis, whereas secondary (slave) ones are condensed out. The primary set is chosen to describe the lowest eigenmodes accurately.

Since static condensation[1] was introduced for the analysis of eigenproblems, considerable progress has been made in an effort to overcome the limitation of slow convergence. A more accurate recovery of the secondary set was made possible using the frequency-dependent transformation equation.[2–4]

Many research studies have focused on the problem of which and how many degrees of freedom should be included in the primary set.[5–8] Usually, the sequential elimination of the secondary degrees of freedom is based on the lowest ratio of diagonal terms of the structural matrices (m_{ii}/k_{ii}). As an alternative approach, the energy method that employs the energy distribution matrix can provide an effective guideline for the selection of the primary set. Linear independence[9–11] of the primary degrees of freedom is used for the sensor placement in dynamic testing.

Dynamic condensation[12–14] shifts the eigenproblem toward the modes of interest, and the mass effect is taken into account in the system transformation. Hence, the condensation error is decreased, and good solutions are obtained for the modes in the neighborhood of the shift point.

In the hybrid dynamic condensation,[15] the static condensation is followed by a modified form of the subspace iteration method. Although the inverse iteration is conducted in a reduced subspace, the formulation is exact and the convergence is accelerated. If several inverse iterations do not give a converged solution, a subspace transformation is employed to orthogonalize the approximate modes. The expansion of the subspace eigenproblem is, in fact, equivalent to the generalized dynamic reduction[16] in MSC/NASTRAN.

The higher-order relation between the primary and secondary degrees of freedom is frequency dependent and, thus, differs from mode to mode, which makes the system transformation very difficult. The transformation is not unique, and there exists a constant matrix that transforms the modes of interest into a reduced subspace yielding the exact eigenvalues of the full system.

In the system equivalent reduction expansion process,[17–18] the reduced eigensystem is solved to get approximate mode shapes as basis vectors. This can be considered a Rayleigh–Ritz method or an application of the eigenvector expansion theorem.

The present study was motivated by the multistep procedures[19–21] employed to improve the system transformation in an iteration procedure. In the series expansion of the transformation matrix, the improved reduction system (IRS) takes terms up to first-order that are approximated using the equation of motion in the reduced subspace. Numerical investigation has led the current authors to include the second-order term in the transformation. A couple of higher-order terms[22] can give a little improvement when a reasonable selection is made for the primary set.

The procedure to be developed here is similar in spirit to the succession-level approximate reduction (SAR), in which the transformation matrix is updated using the reduced equation of an equivalent system. The mass matrix is modified while the stiffness remains constant.

Note that the method of SAR is, in fact, a simplified version of the subspace iteration method. It has turned out that the procedure of transformation update implicitly incorporates not only inverse iteration but also subspace transformation.

The important work of Friswell et al.[21] combines the IRS with the update of the reduced equation of motion. Although the symmetry of the reduced matrices is preserved and the approximate modes can be orthogonalized to some extent, the solution loses accuracy in the lowest modes, which results in slow convergence. The solution needs to be improved through inverse iteration.

The present work establishes a systematic derivation and a comparison of the equations involved in various condensation methods. The basic characteristics will be examined from different points of view. The series expansion of the transformation matrix is equivalent to repeated updates of the matrix through inverse iteration. Through the use of higher-order approximation and system update, both the truncation and condensation errors can be decreased.

Convergence is sensitive to the selection of the primary degrees of freedom. An appropriate set needs to be provided so that the convergence condition may be satisfied; that is, the eigenvalues of the reduced system should be smaller than the lowest eigenvalue of the secondary subspace.

When a poor selection is made, the transformation matrix can be numerically unstable and ill conditioned. Then, repeated updates can make the reduced structural matrices nonpositive definite, which results in a sudden failure of the method of transformation update. In this case, the hybrid dynamic condensation can be employed to get a converging solution.

Once considerable accuracy has been obtained, it is difficult to improve the solution further through subspace transformation. This numerical difficulty is usually encountered in transformation methods because the approximate modes are nearly orthogonal and satisfy the necessary condition of orthogonality for the full system.[23] Then, several inverse iterations may be recommended to push the approximation further to the exact modes.

Static Condensation

In finite element analysis, a general eigenproblem is obtained for undamped free vibration:

$$[k]\{\phi\} = \lambda[m]\{\phi\} \tag{1}$$

where $[k]$ and $[m]$ are the stiffness and mass matrices. An eigenpair is denoted as λ and $\{\phi\}$.

Equation (1) can be expressed in partitioned form as

$$\begin{pmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{pmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \lambda \begin{pmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{pmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} \tag{2}$$

The primary (master) degrees of freedom $\{\phi_p\}$ are included in the analysis, whereas the secondary (slave) set $\{\phi_s\}$ is condensed out.

The relation between the primary and secondary sets is obtained from the second equation of Eq. (2):

$$\{\phi_s\} = -([k_{ss}] - \lambda[m_{ss}])^{-1}([k_{sp}] - \lambda[m_{sp}])\{\phi_p\} \equiv [T(\lambda)]\{\phi_p\} \tag{3}$$

Note that the transformation matrix $[T(\lambda)]$ is frequency dependent.

In Guyan's[1] static condensation, the mass associated with the secondary set is neglected, and the transformation becomes constant for all modes:

$$\{\phi_s\} \cong \{\phi_s\}_{\text{app}} = -[k_{ss}]^{-1}[k_{sp}]\{\phi_p\}_{\text{app}} \equiv [T_0]\{\phi_p\}_{\text{app}} \tag{4}$$

or

$$\{\phi\} \cong \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix}_{\text{app}} = \begin{pmatrix} I \\ T_0 \end{pmatrix} \{\phi_p\}_{\text{app}} \equiv [T_p]\{\phi_p\}_{\text{app}} \tag{5}$$

Substituting Eq. (5) into Eq. (1), one gets a residual error because the transformation is not exact for the original system that has a full mass matrix,

$$\{R\} = [k][T_p]\{\phi_p\}_{\text{app}} - \lambda[m][T_p]\{\phi_p\}_{\text{app}} \tag{6}$$

If the residual weighted by $[T_p]^T$ vanishes, an eigenproblem in the reduced subspace is obtained in terms of the primary degrees of freedom:

$$[K_G]\{\phi_p\}_{\text{app}} = \lambda_{\text{app}}[M_G]\{\phi_p\}_{\text{app}} \tag{7}$$

where the reduced structural matrices are written as

$$\begin{aligned}[K_G] &= [T_p]^T[k][T_p] \\ &= [k_{pp}] + [k_{ps}][T_0] = [k_{pp}] - [k_{ps}][k_{ss}]^{-1}[k_{sp}] \\ [M_G] &= [T_p]^T[m][T_p] \\ &= [m_{pp}] + [m_{ps}][T_0] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}][T_0]\end{aligned} \tag{8}$$

The reduced eigenproblem [Eq. (7)] is solved to get the primary modes $\{\phi_p\}_{\text{app}}$, which are, in turn, used to recover the secondary modes $\{\phi_s\}_{\text{app}}$ in Eq. (4).

Improved Condensation

Increasing attention has been focused on the improvement of the transformation, which is the main source of error in condensation. The matrix inversion in Eq. (3) contains the unknown eigenvalue and is approximated by an infinite series,

$$\begin{aligned}([k_{ss}] - \lambda[m_{ss}])^{-1} &= ([I] - \lambda[A_s])^{-1}[k_{ss}]^{-1} \\ &= \left([I] + \lambda[[A_s] + \lambda^2[A_s]^2 + \cdots\right)[k_{ss}]^{-1}\end{aligned} \tag{9}$$

where

$$[A_s] = [k_{ss}]^{-1}[m_{ss}] \tag{10}$$

As is well known, the series converges in lower modes for which λ is smaller than λ_s, the lowest eigenvalue in the subspace of the secondary degrees of freedom.

Substituting Eq. (9) into Eq. (3) gives

$$\begin{aligned}\{\phi_s\} = &-\left([I] + \lambda[A_s] + \lambda^2[A_s]^2 + \cdots\right)[k_{ss}]^{-1} \\ &\times([k_{sp}] - \lambda[m_{sp}])\{\phi_p\}\end{aligned} \tag{11}$$

When the first-order term is included, the transformation becomes

$$\begin{aligned}\{\phi_s\}_{\text{app}} &= -([I] + \lambda_{\text{app}}[A_s])[k_{ss}]^{-1}([k_{sp}] - \lambda_{\text{app}}[m_{sp}])\{\phi_p\}_{\text{app}} \\ &= \left([T_0] + \lambda_{\text{app}}[k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])\right. \\ &\quad \left. + \lambda_{\text{app}}^2[A_s][k_{ss}]^{-1}[m_{sp}]\right)\{\phi_p\}_{\text{app}}\end{aligned} \tag{12}$$

Using the reduced equation of motion, Eq. (7), one gets

$$(\lambda\{\phi_p\})_{\text{app}} = [M_G]^{-1}[K_G]\{\phi_p\}_{\text{app}} \equiv [D_G]\{\phi_p\}_{\text{app}} \tag{13}$$

Substituting Eq. (13) into Eq. (12) yields

$$\{\phi_s\}_{\text{app}} = ([T_0] + [T_1] + [T_2])\{\phi_p\}_{\text{app}} \tag{14}$$

where

$$\begin{aligned}[T_0] &= -[k_{ss}]^{-1}[k_{sp}], \qquad [T_1] = [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])[D_G] \\ [T_2] &= [A_s][k_{ss}]^{-1}[m_{sp}][D_G]^2\end{aligned} \tag{15}$$

On the other hand, Eq. (11) can be expressed in terms of the eigenvalue:

$$\{\phi_s\} = \big([T_0] + \lambda[k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0]) + \lambda^2[A_s][k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0]) + \lambda^3[A_s]^2[k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0]) + \cdots\big)\{\phi_p\} \tag{16}$$

Hence, one obtains a general formulation for the transformation:

$$\{\phi_s\}_{\text{app}} = ([T_0] + [T_1] + [T_2] + [T_3] + \cdots)\{\phi_p\}_{\text{app}} \tag{17}$$

where

$$[T_0] = -[k_{ss}]^{-1}[k_{sp}], \qquad [T_1] = [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])[D_G]$$
$$[T_2] = [A_s][T_1][D_G], \qquad [T_3] = [A_s][T_2][D_G] \tag{18}$$

In the IRS, the first-order approximation is used to get the transformation as

$$\{\phi_s\}_{\text{app}} = ([T_0] + [T_1])\{\phi_p\}_{\text{app}} \tag{19}$$

or

$$\{\phi_s\}_{\text{app}} = \left([T_p] + \begin{pmatrix} 0 & 0 \\ 0 & k_{ss}^{-1} \end{pmatrix}[m][T_p][D_G]\right)\{\phi_p\}_{\text{app}} \tag{20}$$

Theoretically, the infinite series of Eq. (17) should converge with the ratio of $(\lambda_{\text{app}}/\lambda_s)$. In practice, however, inclusion of higher-order terms does not guarantee monotonic convergence, and sometimes the solution diverges, especially when a poor selection is made for the primary degrees of freedom.

It is believed that the condensation error of the reduced equation (13) tends to accumulate $[T_i]$ in Eq. (18). Numerical examples show that inclusion of the second-order term can give a little improvement when a reasonably good set of primary degrees of freedom is provided.

Transformation Update

The system transformation should differ from mode to mode, as shown in Eq. (3). It means p matrices are required for the transformation of the lowest p modes, which cannot be implemented in practice. In this section, basic characteristics related with the transformation matrix will be closely examined to compare the illustrated condensation methods.

Note that, in each mode, the transformation is not unique and there can be $s \times (p-1)$ matrices satisfying the transformation equation (3). For the lowest p modes, one can define a transformation matrix $[T]$ such that

$$[\Phi_{sp}] = [T][\Phi_{pp}] \tag{21}$$

Then, the exact transformation matrix is obtained as

$$[T] = [\Phi_{sp}][\Phi_{pp}]^{-1} \tag{22}$$

where the columns of $[\Phi_{pp}]$ should be linearly independent to have a nonsingular matrix.

For the p modes, Eq. (2) can be written in compact form as

$$[k_{pp}][\Phi_{pp}] + [k_{ps}][\Phi_{sp}] = ([m_{pp}][\Phi_{pp}] + [m_{ps}][\Phi_{sp}])[\Lambda_p]$$
$$[k_{sp}][\Phi_{pp}] + [k_{ss}][\Phi_{sp}] = ([m_{sp}][\Phi_{pp}] + [m_{ss}][\Phi_{sp}])[\Lambda_p] \tag{23}$$

Substituting Eq. (21) into Eq. (23) gives

$$([k_{pp}] + [k_{ps}][T])[\Phi_{pp}] = ([m_{pp}] + [m_{ps}][T])[\Phi_{pp}][\Lambda_p]$$
$$([k_{sp}] + [k_{ss}][T])[\Phi_{pp}] = ([m_{sp}] + [m_{ss}][T])[\Phi_{pp}][\Lambda_p] \tag{24}$$

Using Eq. (21) and the second equation of Eq. (23), one obtains a basic equation for the transformation:

$$[\Phi_{sp}] = [T_0][\Phi_{pp}] + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T])[\Phi_{pp}][\Lambda_p] \tag{25}$$

Depending on how $[\Phi_{pp}][\Lambda_p]$ is expressed in terms of $[\Phi_{pp}]$, we obtain various transformations. Assume an equivalent system expressed as

$$[K_e][\Phi_{pp}] = [M_e][\Phi_{pp}][\Lambda_p] \tag{26}$$

where $[K_e]$ and $[M_e]$ are the equivalent stiffness and mass matrices. Note that Eq. (26) is similar to the reduced equation of motion [Eq. (7)]. In general, however, the equivalent structural matrices can be nonsymmetric, whereas the reduced structural matrices in Eq. (8) are symmetric.

Multiplying Eq. (25) by $[\Phi_{pp}]^{-1}$ and using Eq. (26) one gets an implicit relation for the transformation matrix $[T]$ in Eq. (22):

$$[T] = [T_0] + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T])[M_e]^{-1}[K_e] \tag{27}$$

Comparing the first equation of Eq. (24) and Eq. (26) gives

$$[K_e] = [k_{pp}] + [k_{ps}][T], \qquad [M_e] = [m_{pp}] + [m_{ps}][T] \tag{28}$$

Substituting Eq. (25) into the first equation of Eq. (23) and comparing the resulting expression with Eq. (26), one obtains

$$[K_e] = [k_{pp}] - [k_{ps}][k_{ss}]^{-1}[k_{sp}] \equiv [K_G]$$
$$[M_e] = [m_{pp}] + [m_{ps}][T] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}][T] \tag{29}$$

On the other hand, the second equation of Eq. (24) is weighted by $[T_0]^T$ and added to the first one to get

$$[K_e] = [k_{pp}] + [k_{ps}][T] + [T_0]^T[k_{sp}] + [T_0]^T[k_{ss}][T]$$
$$[M_e] = [m_{pp}] + [m_{ps}][T] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}][T] \tag{30}$$

Substituting Eq. (27) into the first equation of Eq. (30) gives

$$[K_e] = [k_{pp}] + [k_{ps}][T_0] + [T_0]^T[k_{sp}] + [T_0]^T[k_{ss}][T_0]$$
$$= [k_{pp}] - [k_{ps}][k_{ss}]^{-1}[k_{sp}] \equiv [K_G] \tag{31}$$

Hence, Eq. (30) is the same as Eq. (29).

If $[T]^T$ is used for the weighting, one gets symmetric matrices as

$$[K_e] = [k_{pp}] + [k_{ps}][T] + [T]^T[k_{sp}] + [T]^T[k_{ss}][T]$$
$$[M_e] = [m_{pp}] + [m_{ps}][T] + [T]^T[m_{sp}] + [T]^T[m_{ss}][T] \tag{32}$$

When the exact transformation of Eq. (22) is used, the equivalent structural matrices in Eqs. (28–32) should satisfy Eq. (26) exactly. In practice, however, the exact matrix is not known, and iterative methods are used to improve approximations for the transformation matrix of Eq. (27).

As the iterative procedure continues, the transformation matrix should converge monotonically to the exact one of Eq. (22). Also, it would be more reasonable to employ the same matrix for both transformation and weighting, which yields symmetric equivalent matrices as in the static condensation. Therefore, it would be expected that the system transformation with Eq. (32) or the method of Friswell et al.[21] would be favored over Eq. (29) or the SAR method.

In numerical investigations, however, the solution iteration of Friswell et al.[21] loses accuracy in the lowest modes, which results in slow convergence for higher modes. On the other hand, the method of SAR shows steady and faster convergence. Equation (28) represents partial (p out of $p+s$) equations in Eq. (24) and, hence, is not complete. Iterations with the nonsymmetric matrices give diverging results.

In fact, the transformation update of SAR contains the procedure of inverse iteration and, hence, preserves the accuracy in the lowest modes. In addition, the subspace transformation is not necessary to orthogonalize the approximate modes.

In the following sections, the equations involved in various condensation methods will be elaborated from different points of view. Also, basic formulations will be examined in greater detail to clarify how the transformation matrix can be derived and improved through iterative updates.

Hybrid Dynamic Condensation

In system condensation, a proper set of primary degrees of freedom should be provided so that the convergence condition may be satisfied. When a poor selection is made for the primary set, the transformation matrix can be ill conditioned and there is a real possibility of solution divergence.

In the hybrid dynamic condensation,[15] which is a modified form of the subspace iteration method, the static condensation is followed by inverse iterations and subspace transformation. The inverse iteration drives the trial vectors toward the lowest modes and accelerates the convergence whereas the subspace transformation orthogonalizes the approximate modes and stabilizes the solution search.

For the lowest p modes, the procedure of inverse iteration is written as

$$[k][X_p] = [m][\Phi_p]_{\text{app}} \tag{33}$$

or

$$\begin{pmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{pmatrix} \begin{pmatrix} X_{pp} \\ X_{sp} \end{pmatrix} = \begin{pmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{pmatrix} \begin{pmatrix} \Phi_{pp} \\ \Phi_{sp} \end{pmatrix}_{\text{app}} \tag{34}$$

The relation between the primary and secondary sets is approximated by

$$[\Phi_{sp}]_{\text{app}} = [T_r][\Phi_{pp}]_{\text{app}} \tag{35}$$

where $[T_r]$ is an approximate transformation matrix. Then, Eq. (33) is reduced to get an equivalent system written as

$$[K_G][X_{pp}] = [M_e][\Phi_{pp}]_{\text{app}} \tag{36}$$

The equivalent mass matrix is obtained as

$$[M_e] = [m_{pp}] + [m_{ps}][T_r] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}][T_r] \tag{37}$$

Inasmuch as it is conducted in a reduced subspace as shown in Eq. (36), the inverse calculation for $[\Phi_p]_{\text{app}}$ is exact. Equation (37) is similar to the equivalent mass of Eq. (29).

The secondary part is recovered as

$$[X_{sp}] = [T_0][X_{pp}] + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_r])[\Phi_{pp}]_{\text{app}} \tag{38}$$

Using Eq. (36), Eq. (38) can be rewritten as

$$[X_{sp}] = [T_X][X_{pp}] \tag{39}$$

where

$$[T_X] = [T_0] + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_r])[M_e]^{-1}[K_G] \tag{40}$$

The matrix $[T_X]$ shows the relation between the primary and secondary parts of the approximate modes $[X_p]$ improved through inverse iteration. Note that the improved transformation matrix of Eq. (40) can be compared with Eq. (27).

Comparison of Eqs. (35) and (39) shows that the procedure of inverse iteration is equivalent to the transformation update. It has been found that repeated updates of the equivalent mass and transformation matrices are, in fact, the same as in the method of SAR, which will be explained in the next section.

The improved modes are written as

$$[X_p] = \begin{pmatrix} X_{pp} \\ X_{sp} \end{pmatrix} = \begin{pmatrix} I \\ T_X \end{pmatrix} [X_{pp}] \tag{41}$$

As is well known, the inverse calculation pushes the iteration vectors further to the exact eigenmodes simultaneously, and hence, the approximation becomes rich in the lowest modes. Hence, $[X_p]$ is usually not orthogonal with respect to the original system. The loss of orthogonality may cause slow convergence in higher modes.

To orthogonalize and separate the approximate modes, the subspace transformation can be implemented in practice. In fact, the Rayleigh quotient is minimized for an arbitrary combination of the approximate vectors. Several inverse iterations are recommended before subspace transformation.

An orthogonal set of approximate modes are obtained as

$$[\Psi_p] = [X_p][Q_{pp}] = \begin{pmatrix} I \\ T_X \end{pmatrix} [X_{pp}][Q_{pp}] = \begin{pmatrix} I \\ T_X \end{pmatrix} [\Psi_{pp}] \tag{42}$$

where $[Q_{pp}]$ and $[\Psi_{pp}]$ are the eigensolutions of the reduced systems or

$$[X_{pp}]^T[K_X][X_{pp}][Q_{pp}] = [X_{pp}]^T[M_X][X_{pp}][Q_{pp}][\Lambda_p] \tag{43}$$

$$[K_X][\Psi_{pp}] = [M_X][\Psi_{pp}][\Lambda_p] \tag{44}$$

The reduced structural matrices are written as

$$\begin{aligned} [K_X] &= [k_{pp}] + [k_{ps}][T_X] + [T_X]^T[k_{sp}] + [T_X]^T[k_{ss}][T_X] \\ [M_X] &= [m_{pp}] + [m_{ps}][T_X] + [T_X]^T[m_{sp}] + [T_X]^T[m_{ss}][T_X] \end{aligned} \tag{45}$$

Instead of $[X_p]$ or $[X_{pp}]$ of Eq. (41), the orthogonal modes $[\Psi_p]$ or $[\Psi_{pp}]$ in Eq. (42) are substituted into Eq. (33) or (36) for inverse iteration. Note that the same transformation matrix $[T_X]$ is used for Eqs. (41) and (42).

Note that neither $[X_{pp}]$ nor $[\Psi_{pp}]$ is related directly with the update of the equivalent mass matrix. Only the transformation matrix $[T_r]$ is used to obtain the mass matrix in Eq. (37) and the reduced equation for the equivalent system [Eq. (36)].

On convergence, the reduced equation of motion in Eq. (44) is solved to get $[\Psi_{pp}]$ and $[\Lambda_p]$. In intermediate steps of iteration, it may be assumed that the orthogonalization is included implicitly in the procedure of matrix update. This is one of the most important advantages of the iterative method for condensation.

Iterative Condensation

In the SAR method,[20] the system condensation is improved in iterations. The approach can be considered as a procedure of inverse iteration:

$$[k]\{\phi^{(i+1)}\} = \lambda^{(i)}[m]\{\phi^{(i)}\} \tag{46}$$

which is rewritten in separate form as

$$\begin{aligned} &[k_{pp}]\{\phi_p^{(i+1)}\} + [k_{ps}]\{\phi_s^{(i+1)}\} \\ &\quad = [m_{pp}]\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) + [m_{ps}]\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) \\ &\quad = \left([m_{pp}] + [m_{ps}]\left[T_r^{(i)}\right]\right)\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) \\ &[k_{sp}]\{\phi_p^{(i+1)}\} + [k_{ss}]\{\phi_s^{(i+1)}\} \\ &\quad = [m_{sp}]\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) + [m_{ss}]\left(\lambda^{(i)}\{\phi_s^{(i)}\}\right) \\ &\quad = \left([m_{sp}] + [m_{ss}]\left[T_r^{(i)}\right]\right)\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) \end{aligned} \tag{47}$$

The transformation relation is expressed as

$$\{\phi_s^{(i)}\} = \left[T_r^{(i)}\right]\{\phi_p^{(i)}\} \tag{48}$$

and, thus, the second equation of Eq. (47) can be rewritten as

$$\begin{aligned} \{\phi_s^{(i+1)}\} &= -[k_{ss}]^{-1}[k_{sp}]\{\phi_p^{(i+1)}\} + [k_{ss}]^{-1} \\ &\quad + \left([m_{sp}] + [m_{ss}]\left[T_r^{(i)}\right]\right)\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) \end{aligned} \tag{49}$$

Substitution of Eq. (49) into the first equation of Eq. (47) gives

$$[K_G]\{\phi_p^{(i+1)}\} = \left[M_e^{(i)}\right]\left(\lambda^{(i)}\{\phi_p^{(i)}\}\right) \tag{50}$$

where the equivalent mass matrix is obtained as

$$\left[M_e^{(i)}\right] = [m_{pp}] + [m_{ps}]\left[T_r^{(i)}\right] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}]\left[T_r^{(i)}\right] \tag{51}$$

From Eqs. (49) and (50), one gets

$$\{\phi_s^{(i+1)}\} = \left[T_r^{(i+1)}\right]\{\phi_p^{(i+1)}\} \tag{52}$$

where the updated transformation matrix is

$$\left[T_r^{(i+1)}\right] = [T_0] + [k_{ss}]^{-1}\left([m_{sp}] + [m_{ss}]\left[T_r^{(i)}\right]\right)\left[M_e^{(i)}\right]^{-1}[K_G] \tag{53}$$

For easy understanding and comparison with the IRS, the sequential calculation can be illustrated as follows. At the initial step, the static transformation of Eq. (4) is used. Then Eq. (49) becomes

$$\left\{\phi_s^{(1)}\right\} = -[k_{ss}]^{-1}[k_{sp}]\left\{\phi_p^{(1)}\right\} + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])\left(\lambda^{(0)}\left\{\phi_p^0\right\}\right) \tag{54}$$

The equation of the equivalent system is obtained as

$$[K_G]\left\{\phi_p^{(1)}\right\} = [M_G]\left(\lambda^{(0)}\left\{\phi_p^{(0)}\right\}\right) \tag{55}$$

From Eqs. (54) and (55), one gets

$$\left\{\phi_s^{(1)}\right\} = ([T_0] + [T_1])\left\{\phi_p^{(1)}\right\} \equiv \left[T_r^{(1)}\right]\left\{\phi_p^{(1)}\right\} \tag{56}$$

where

$$[T_1] = [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])[M_G]^{-1}[K_G] \tag{57}$$

The relation is same as in the IRS.

In the same way, the second iteration shows

$$\left\{\phi_s^{(2)}\right\} = -[k_{ss}]^{-1}[k_{sp}]\left\{\phi_p^{(2)}\right\} + [k_{ss}]^{-1}([m_{sp}] + [m_{ss}]([T_0] + [T_1]))\left(\lambda^{(1)}\left\{\phi_p^{(1)}\right\}\right) \tag{58}$$

Assuming the same reduced equation (55), one gets

$$\left\{\phi_s^{(2)}\right\} = ([T_0] + [T_1] + [T_2])\left\{\phi_p^{(2)}\right\} \equiv \left[T_r^{(2)}\right]\left\{\phi_p^{(2)}\right\} \tag{59}$$

where

$$[T_2] = [A_s][T_1][M_G]^{-1}[K_G] \tag{60}$$

Hence, the second-order term is added. As the iteration continues, the higher-order terms are included as in Eq. (17).

If the reduced equation of motion in the static condensation is used for the equivalent system, the repeated update of the transformation matrix is the same as the addition of higher-order terms in the series expansion of the matrix. To prevent the condensation error from accumulating, the equivalent mass matrix and the relevant equation for the equivalent system should be updated at each iteration.

Accelerated Condensation

When a reasonably good selection is made for the primary degrees of freedom, inclusion of the second-order term in the series expansion of the transformation matrix may show a little improvement for the solution convergence. Hence, the initial transformation associated with the static condensation is assumed as

$$\left[T_r^{(1)}\right] = [T_0] + [T_1] + [T_2] \tag{61}$$

where

$$[T_0] = -[k_{ss}]^{-1}[k_{sp}], \qquad [T_1] = [k_{ss}]^{-1}([m_{sp}] + [m_{ss}][T_0])[D_G]$$
$$[T_2] = [A_s][T_1][D_G] \tag{62}$$

The transformation matrix is updated through inverse iteration as

$$\left[T_r^{(i+1)}\right] = [T_0] + [k_{ss}]^{-1}\left([m_{sp}] + [m_{ss}]\left[T_r^{(i)}\right]\right)\left[M_e^{(i)}\right]^{-1}[K_G] \tag{63}$$

where the equivalent mass matrix is

$$\left[M_e^{(i)}\right] = [m_{pp}] + [m_{ps}]\left[T_r^{(i)}\right] + [T_0]^T[m_{sp}] + [T_0]^T[m_{ss}]\left[T_r^{(i)}\right] \tag{64}$$

As a convergence criterion, the rate of eigenvalue change can be used for the modes of interest,

$$\left|\left(\lambda_j^{(i+1)} - \lambda_j^{(i)}\right)\middle|\ \lambda_j^{(i)}\right| < \varepsilon, \qquad j \le p \tag{65}$$

where ε is a specified tolerance.

To get the eigenvalues, the reduced eigenproblem is written as

$$\left[K_r^{(i+1)}\right]\left[\Psi_{pp}^{(i+1)}\right] = \left[M_r^{(i+1)}\right]\left[\Psi_{pp}^{(i+1)}\right]\left[\Lambda_{pp}^{(i+1)}\right] \tag{66}$$

where the reduced structural matrices are obtained as

$$\left[K_r^{(i+1)}\right] = [k_{pp}] + [k_{ps}]\left[T_r^{(i+1)}\right] + \left[T_r^{(i+1)}\right]^T[k_{sp}] + \left[T_r^{(i+1)}\right]^T[k_{ss}]\left[T_r^{(i+1)}\right]$$
$$\left[M_r^{(i+1)}\right] = [m_{pp}] + [m_{ps}]\left[T_r^{(i+1)}\right] + \left[T_r^{(i+1)}\right]^T[m_{sp}] + \left[T_r^{(i+1)}\right]^T[m_{ss}]\left[T_r^{(i+1)}\right] \tag{67}$$

It is a burden to solve the eigenproblem of Eq. (66) in each iteration. Therefore, the convergence check may be carried out at every second or third iteration. In practice, there is little chance that the solution converges in a few iterations, and hence the test may be skipped on the initial stage of the solution.

When the selection of the primary degrees of freedom is inappropriate, so is the static transformation matrix $[T_0]$. Then, the equivalent mass matrix $[M_e^{(i)}]$ may be ill conditioned and no longer positive definite. In this case, the updated matrix $[T_r^{(i+1)}]$ does not converge to the exact one in Eq. (22).

In addition, the inclusion of $[T_2]$ leads to a devastating effect on the system transformation. Direct application of the hybrid dynamic condensation is preferred over the method of transformation update.

The equation for inverse iteration can be written in partitioned form as

$$\begin{pmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{pmatrix}\begin{pmatrix} X_{pp} \\ X_{sp} \end{pmatrix} = \begin{pmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{pmatrix}\begin{pmatrix} \Phi_{pp} \\ \Phi_{sp} \end{pmatrix}_{\text{app}} = \begin{pmatrix} B_{pp} \\ B_{sp} \end{pmatrix} \tag{68}$$

which can be reduced to get

$$[K_G][X_{pp}] = [B_{pp}] + [T_0]^T[B_{sp}] \tag{69}$$

The secondary set is recovered as

$$[X_{sp}] = [T_0][X_{pp}] + [k_{ss}]^{-1}[B_{sp}] \tag{70}$$

After several inverse iterations, the subspace transformation is conducted to get a set of orthogonal approximate modes:

$$[\Psi_p] = [X_p][Q_{pp}] \tag{71}$$

where $[Q_{pp}]$ is the eigensolution of the transformed system

$$[X_p]^T[k][X_p][Q_{pp}] = [X_p]^T[m][X_p][Q_{pp}][\Lambda_p] \tag{72}$$

Numerical Examples

L-Shaped Beam

The flexural vibration of a uniform, L-shaped beam[14] is used to examine the numerical performance of the proposed method. The finite element model has six nodes and five elements as shown in Fig. 1. The bending motion will be described using the nodal displacement with 15 degrees of freedom. A finite element program was used to get the eigenvalues in Table 1.

The primary set includes five degrees of freedom 1, 4, 7, 11, and 14, which are obtained after sequential elimination of the secondary one. Figure 2 shows the truncation effect of the higher-order terms in Eq. (17). The relative errors in the eigenvalue are calculated using the exact values of the full system. The first-order approximation gives an excellent solution for the first three modes. In higher modes, the addition of $[T_2]$ shows a little improvement.

In Fig. 3, the convergence characteristics of various condensation methods are shown for the fifth mode, the highest of the primary modes. In the methods of Friswell et al.[21] and SAR, the first-order approximation is used, whereas the second-order term is included in the accelerated scheme.

Table 1 Eigenvalues of L-shaped beam

Mode	Eigenvalue
1	8.93251×10^2
2	9.48252×10^3
3	1.43525×10^5
4	5.55616×10^5
5	1.27221×10^6
6	5.05878×10^6
7	8.19573×10^6

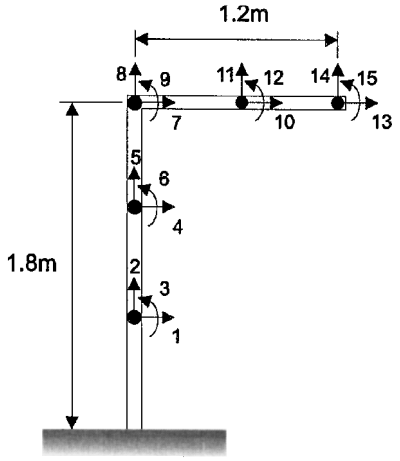

Fig. 1 L-shaped beam in flexural vibration: $A = 1.6 \times 10^3$ mm^2, $I = 2.1333 \times 10^5$ mm^4, $E = 2.1 \times 10^5$ N/mm^2, and $\rho = 7.83 \times 10^{-9}$ N s^2/mm^4.

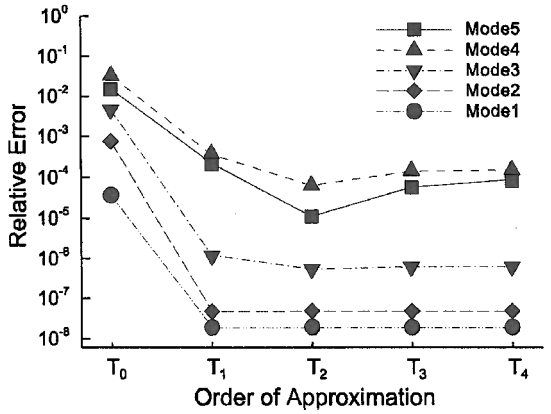

Fig. 2 Truncation effect in series expansion.

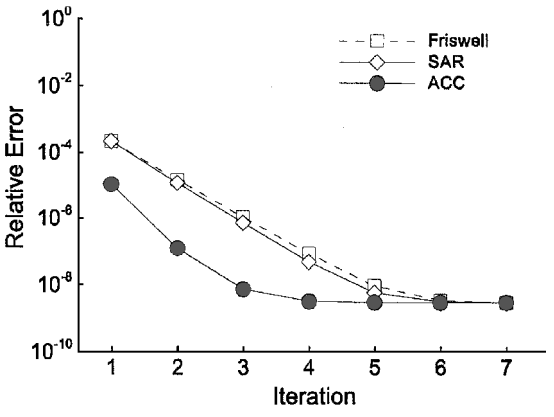

Fig. 3 Convergence of fifth eigenvalue of L-beam.

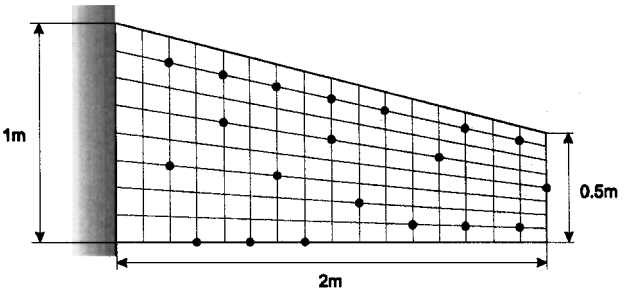

Fig. 4 Wing plate in transverse vibration.

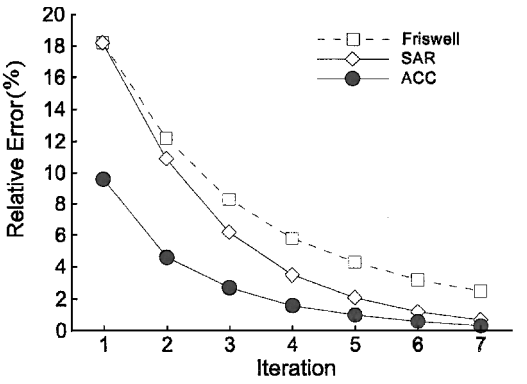

Fig. 5 Convergence for 18th eigenvalue of wing plate.

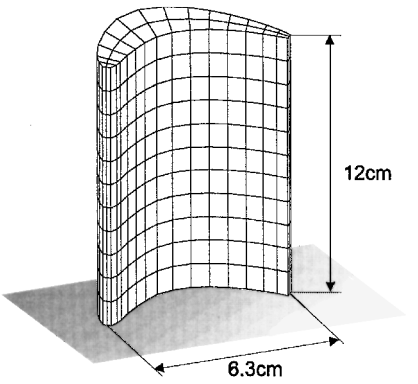

Fig. 6 Turbine blade.

Wing Plate

As an example of intermediate-size systems, the transverse vibration of a wing plate in Fig. 4 is considered. A total of 153 nodes and 128 plate elements are used for the finite element modeling. The points with circular dots indicate 20 primary nodes included in the analysis.

Figure 5 shows the percentage errors obtained in the solution iteration. The convergence for the 18th mode, which has the largest magnitude of error, is illustrated. It is believed that the primary set is better suited for the 19th and 20th modes. The same tendency as in the L-beam can be observed.

Turbine Blade

The finite element model of a turbine blade in Fig. 6 has 308 solid elements and 540 nodes. The blade is constrained at the bottom and contains a total of 1485 translational degrees of freedom. Through sequential elimination, 30 nodes on the midsurface are selected for the primary set, as shown in Fig. 7. The 30th mode has the largest magnitude of percentage errors, shown in Fig. 8.

Effect of Poor Selection of Primary Degrees of Freedom

To investigate the effect of poor selection, a set of primary degrees of freedom 1, 4, 7, 10, and 13 was considered for the L-shaped beam in Fig. 1, and the results are shown in Fig. 9. Whereas direct

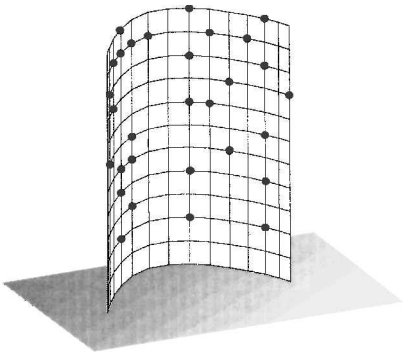

Fig. 7 Primary nodes of turbine blade.

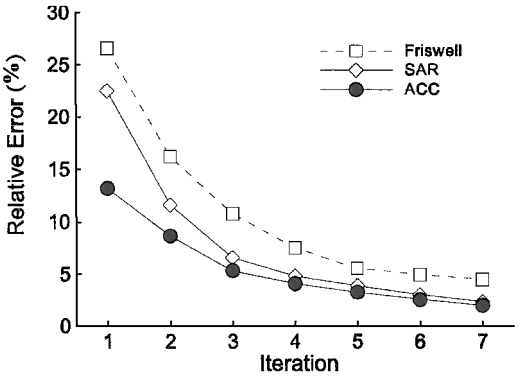

Fig. 8 Convergence of 30th eigenvalue of turbine blade.

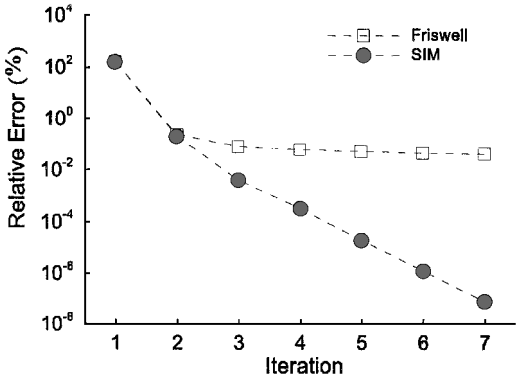

Fig. 9 Effect of poor selection on fifth eigenvalue of L-beam.

application of the hybrid dynamic condensation and the subspace iteration method (SIM) gives steady convergence for the fifth eigenvalue, the method of Friswell et al.[21] with the first-order approximation shows very slow convergence. On the other hand, the application of SAR and the addition of the second-order term cause nonpositive definite mass matrices after several iterations. It is evident that good selection of the primary degrees of freedom is a prerequisite to get a convergent solution in the iterative method of transformation update.

Conclusions

The solution convergence of iterative methods for system condensation has been accelerated through the inclusion of the second-order term in the series expansion of the transformation matrix. Also, the systematic derivation and comparison of the equations involved in various condensations was carried out in the present study.

It has been found that the iterative method of the SAR is closely related to the hybrid dynamic condensation and the SIM. The repeated update of the transformation matrix incorporates not only inverse iteration but also subspace transformation implicitly. A reasonably good selection of the primary degrees of freedom should be made so that the convergence condition can be satisfied.

A poor selection of the primary set can give an inappropriate transformation, which makes the equivalent mass matrix ill conditioned and nonpositive definite. Then, slow convergence or even divergence can be observed in the solution iteration. In this case, direct application of the hybrid dynamic condensation is preferred over the method of transformation update.

Further study is required to define an efficient criterion for convergence. The numerical condition of the transformation matrix and the equivalent mass needs to be properly checked in the iteration process. The current method of higher-order approximation is being extended to the design sensitivity analysis for structural optimization of large systems.

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